Numerical methods in matrix models:
A case study: The GW model

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Matrix models, quantum geometry & beyond

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Outline

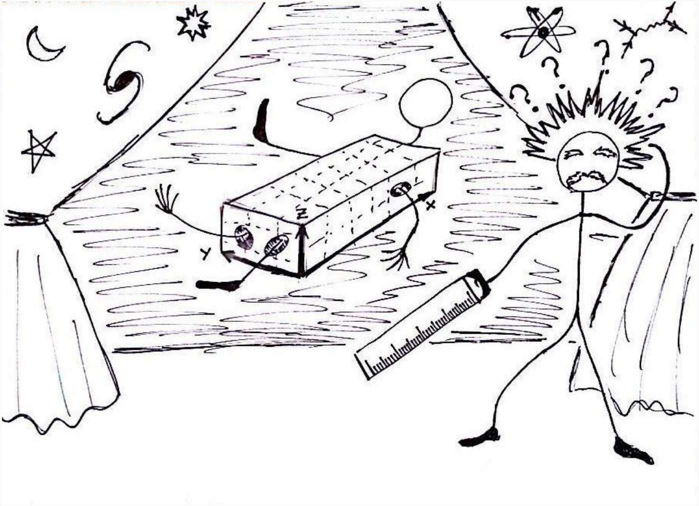
1. GW model
2. Matrix GW model & Phase transitions
3. Distribution hunt
4. Beyond GW: A gauge model
5. Phase/renormalizability connection

Literature

- B. Ydri, Lectures on Matrix Field Theory (arXiv:1603.00924)
- J. Tekel, Phase structure of fuzzy field theories and multitrace matrix models (arXiv:1512.00689)

GW model

- intuition — mini black holes hiding the points we wish to probe
- magnetic field (Girvin, Jach '84); rotating universe? (Shamir '25)



- NC coordinates:

$$[\hat{x}^\mu, \hat{x}^\nu] = \hat{x}^\mu \hat{x}^\nu - \hat{x}^\nu \hat{x}^\mu = i\hat{\theta}^{\mu\nu}, \qquad \Delta\hat{x}^\mu \Delta\hat{x}^\nu \geq \frac{\left|\left\langle \hat{\theta}^{\mu\nu} \right\rangle\right|}{2}$$

- Moyal $\star$-product:

$$f \star g = f e^{i/_2 \overleftarrow{\partial}\theta\overrightarrow{\partial}} g = fg + \frac{i\theta^{\mu\nu}}{2}\frac{\partial f}{\partial x^\mu}\frac{\partial g}{\partial x^\nu} + \cdots \quad \Rightarrow \quad [x^\mu, x^\nu]_\star = i\theta^{\mu\nu}$$

- the simplest model $\lambda\phi^4_\star$:

$$S = \int d^n x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \right)$$

$$f \star g \neq g \star f \qquad (f \star g) \star h = f \star (g \star h)$$

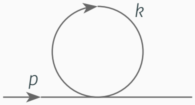
$$\int d^n x\, f \star g = \int d^n x\, fg$$

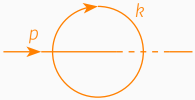
$$\int d^n x\, f_1 \star f_2 \star \cdots \star f_n = \int d^n x\, f_n \star f_1 \star \cdots \star f_{n-1}$$

$$\partial_\mu (f \star g) = \partial_\mu f \star g + f \star \partial_\mu g$$

$$e^{ipx} \star e^{iqx} = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu} e^{i(p+q)x}$$

- propagator/mass corrections in the S_{eff}: $\sim \phi(p^2 + m^2 + \cdots)\phi$





$$\text{planar} = \frac{\lambda}{4!}\int \frac{d^2k}{(2\pi)^2}\frac{2}{k^2 + m^2}$$

$$\text{non-planar} = \frac{\lambda}{4!}\int \frac{d^2k}{(2\pi)^2}\frac{\exp\left(ik_\mu\theta^{\mu\nu}p_\nu\right)}{k^2 + m^2} = \frac{\lambda}{96\pi}\log\frac{\Lambda_{\text{eff}}^2}{m^2} + \cdots$$

$$\Lambda_{\text{eff}}^2 = \frac{1}{1/\Lambda_{\max}^2 + |\theta^{\mu\nu}p_\nu|^2} \to \frac{1}{|\theta^{\mu\nu}p_\nu|^2}, \quad \Lambda_{\max} \to \infty$$

- 1PI 2-point function

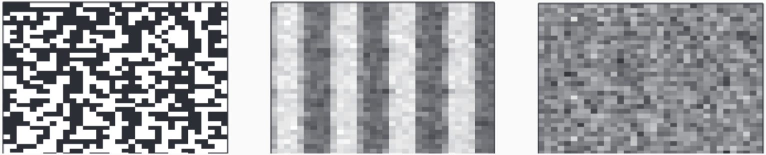
$$\Gamma^{(2)}(p) = p^2 + m_{\rm ren}^2 - \#\lambda \log \tilde{p}^2 + \cdots, \qquad \tilde{p}_\mu = \theta_{\mu\nu} p^\nu$$

- minimized by

$$p_*^2 \sim \lambda$$

- the simplest stripe phase

$$\phi(x) \sim \cos(p_* \cdot x)$$



- center plot represents the stripe phase in coordinate space (Mejía-Díaz, Bietenholz and Panero ’14)
- coordination of the renormalization procedure for several (infinitely many?) particles with different masses
- looks different on different scales

$$S_{GW} = \int d^{2n}x \left(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi + \right.$$
$$\left. + \frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) \right)$$

$$\text{where} \quad [x^\mu, x^\nu]_\star = i\theta^{\mu\nu} \quad \text{and} \quad \tilde{x}_\mu = 2\left(\theta^{-1}\right)_{\mu\nu} x^\nu$$

- superrenormalizable in 2-dim (Grosse and Wulkenhaar ’03)
- renormalizability of the 2-dim $\lambda\phi^4_\star$-model is restored by defining it as a $\Omega \to 0$ limit of the series of GW models in which Ω itself does not renormalize and serves as a series-label

$$S_{GW} = \int d^{2n}x \left(\frac{1}{2}\partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2}\phi \star \phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi + \right.$$
$$\left. + \frac{1}{2}\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) \right)$$

- LS duality (Langmann, Szabo '02), UV $\leftrightarrow$ IR

$$\widehat{\phi}(p) \longleftrightarrow \pi^2 \sqrt{|\det \theta|}\, \phi(x), \qquad p_\mu \longleftrightarrow \tilde{x}_\mu$$

$$S[\phi; m, \lambda, \Omega] \longmapsto \Omega^2\, S\left[\phi; \frac{m}{\Omega}, \frac{\lambda}{\Omega^2}, \frac{1}{\Omega}\right]$$

- harm. osc. potential $\Longleftrightarrow$ curvature (Burić and Wohlgenannt '10, Prekrat '26)

- absorbing the NC length scale, switching to matrices:

$$\left[x^{\mu}, x^{\nu}\right]_{\star} = i\theta^{\mu\nu} \qquad \Longrightarrow \qquad [X, Y] = i\,\mathbb{1}$$

- infinite-dimensional representation: $\quad X \to + \qquad Y/i \to -$

$$\frac{1}{\sqrt{2}}\begin{bmatrix} & \pm 1 & & & \\ +1 & & \pm\sqrt{2} & & \\ & +\sqrt{2} & & \pm\sqrt{3} & \\ & & +\sqrt{3} & & \cdots \\ & & & \vdots & \end{bmatrix}$$

$$[X,Y] = i\epsilon\,(\mathbb{1} - Z) \qquad [X,Z] = i\epsilon\,\{Y,Z\} \qquad [Y,Z] = -i\epsilon\,\{X,Z\}$$

- possible to define diff. calculus (Burić and Wohlgenannt ’10)
- derivatives given by NC coordinates:

$$\partial_\mu = [iP_\mu, \cdot\,] \qquad \epsilon P_1 = Y \qquad \epsilon P_2 = -X \qquad \epsilon P_3 = Z - \frac{\mathbb{1}}{2}$$

- price — space is now curved with curvature:

$$R = \frac{15}{2}\mathbb{1} - 4Z - 8\left(X^2 + Y^2\right) \approx -16\,\mathrm{diag}(1, 2, \ldots, N)$$

Matrix GW model & Phase transitions

- Weyl transform of the GW model:

$$\phi \longleftrightarrow \Phi \qquad\qquad \int \longleftrightarrow \sqrt{\det 2\pi\theta}\ \mathrm{tr}$$

- matrix model on the truncated Heisenberg algebra:

$$\boxed{S_N = N\,\mathrm{tr}\Big(\Phi\widehat{K}\Phi - g_r R\Phi^2 - g_2\Phi^2 + g_4\Phi^4\Big)}$$

- kinetic operator: $\widehat{K} = [X,[X,\cdot\,]] + [Y,[Y,\cdot\,]]$
- unscaled (G's) vs. scaled (g's) parameters:

$$G_2 = Ng_2 \qquad\qquad G_4 = Ng_4$$

$$X \to \frac{X}{\sqrt{N}} \qquad\qquad Y \to \frac{Y}{\sqrt{N}} \qquad\qquad R \to \frac{R}{N}$$

- EOM:

$$2\hat{K}\Phi - g_r\left(R\Phi + \Phi R\right) + 2\Phi\left(2g_4\Phi^2 - g_2\,\mathbb{1}\right) = \mathbb{0}$$

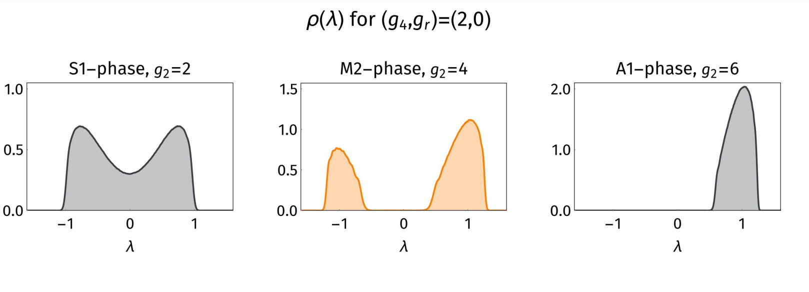
- classical vacuum solutions:

$$\Phi = \frac{\operatorname{tr}\Phi}{N}\,\mathbb{1} \qquad \Phi = \mathbb{0} \qquad \Phi^2 = \begin{cases} \mathbb{0} & \text{for } g_2 \leq 0 \\ \frac{g_2\mathbb{1}}{2g_4} & \text{for } g_2 > 0 \end{cases}$$

- 3 phases:
  - disordered/symmetric 1-cut phase (S1): $\Phi_{S1} = \mathbb{0}$
  - striped/2-cut phase (M2, S2): $\Phi_{M2} \propto U\,\mathbb{1}_{\pm}U^{\dagger}$
  - ordered/asymmetric 1-cut phase (A1): $\Phi_{A1} \propto \mathbb{1}$

- modified ordered phases (and other beasts):

$$\Phi_R^2 = \frac{g_2\mathbb{1} - g_r|R|}{2g_4}$$



- disordered/symmetric 1-cut phase (S1): $\Phi_{S1} = \mathbb{0}$
- **striped/2-cut phase (M2, S2):** $\Phi_{M2} \propto U\,\mathbb{1}_{\pm}U^{\dagger}$
- ordered/asymmetric 1-cut phase (A1): $\Phi_{A1} \propto \mathbb{1}$

- we are interested in:

$$\langle O \rangle = \frac{\int [d\Phi]\, O\, e^{-S_N}}{\int [d\Phi]\, e^{-S_N}} \qquad \mathrm{Var}\, O = \langle O^2 \rangle - \langle O \rangle^2$$

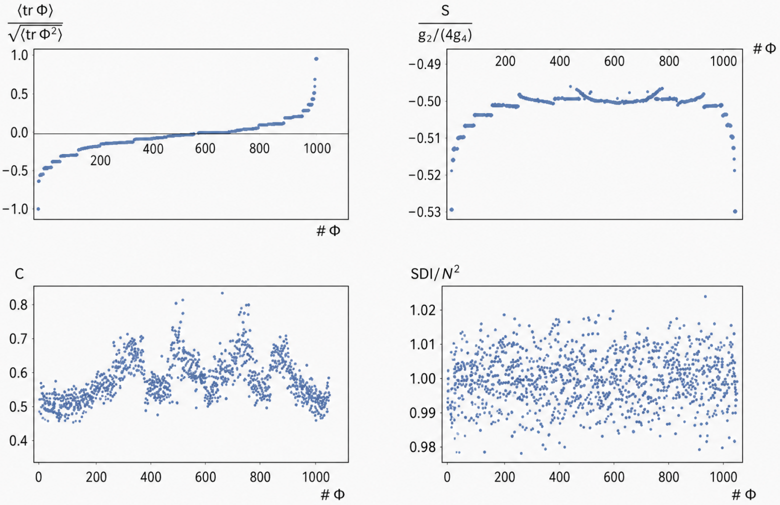
- can be computed numerically (Hamiltonian Monte Carlo)

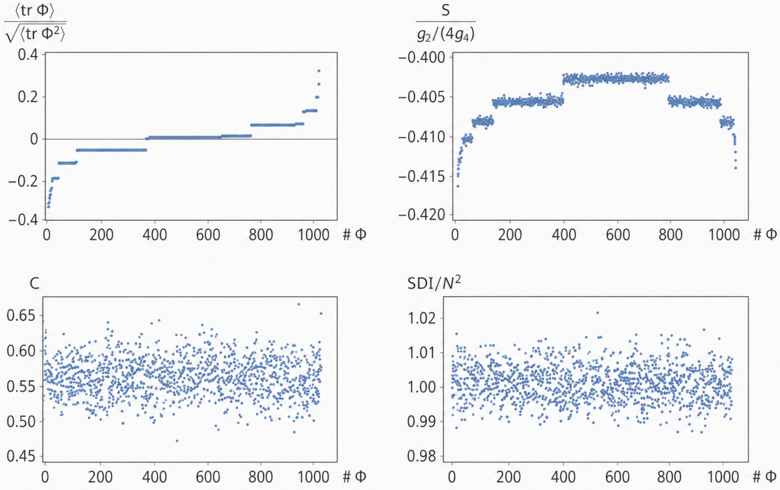
- typical thermodynamic quantities:
  - heat capacity $C = \mathrm{Var}\, S/N^2$
  - susceptibility $\chi = \mathrm{Var}\, |\operatorname{tr} \Phi|/N^2$

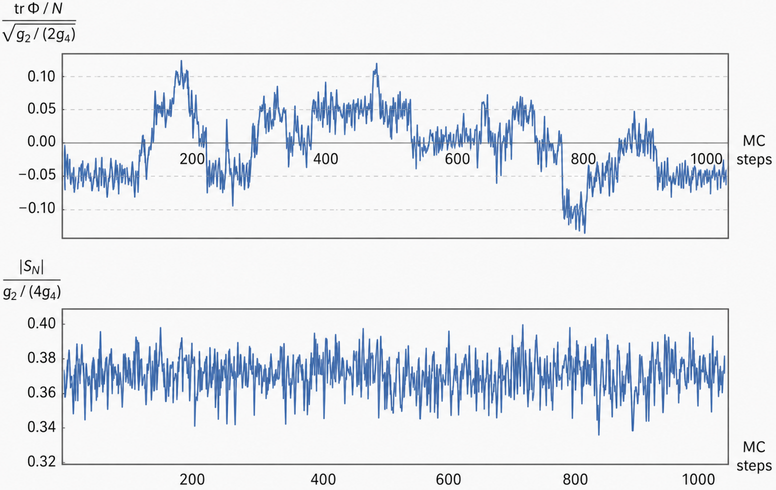
- random field characteristics:
  - eigenvalue (ρ_λ) and trace (ρ_{tr}) probability distributions

- fuzzy sphere, fuzzy onion (Bratislava: Kováčik, Tekel, Hrmo, Rusnák), bootstrap...
- baseline: pure potential model (PP)
- Paradox: PP matrix model does not see the details of NC unlike its $\star$-product version? Or is it...

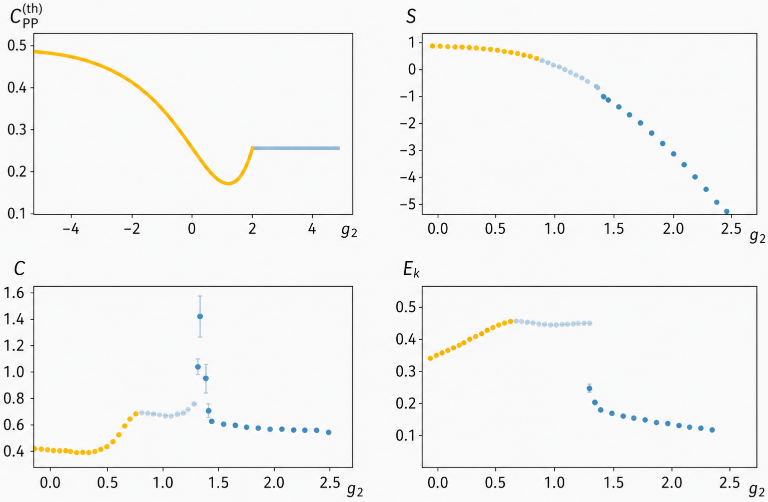
Different HMC start configurations (A1 phase, $N = 24$)

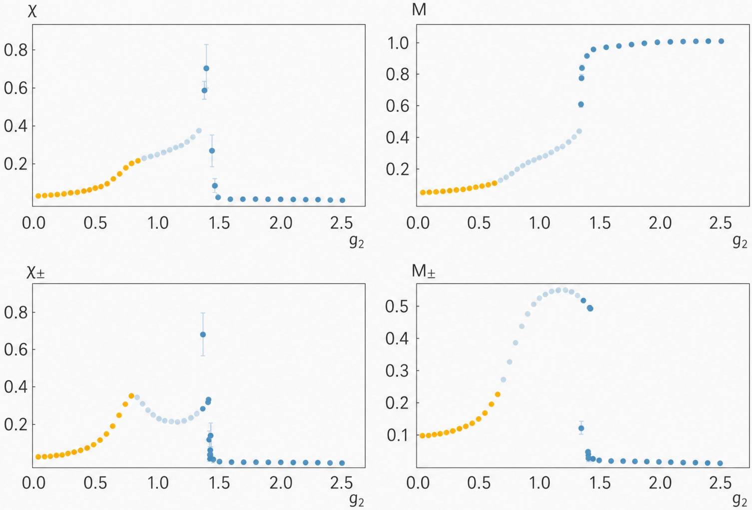


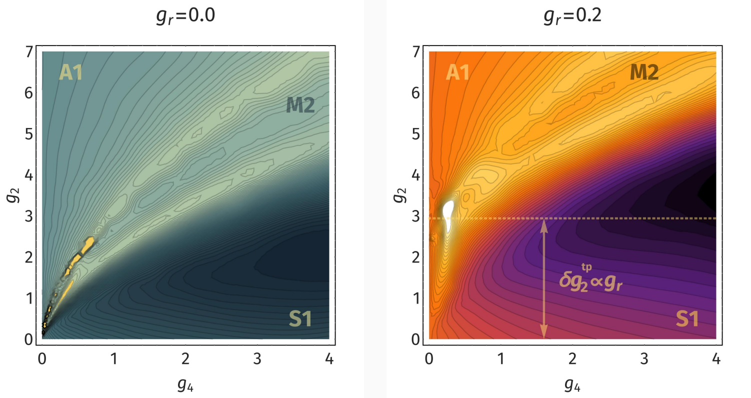

$\frac{\langle \mathrm{tr}\,\Phi \rangle}{\sqrt{\langle \mathrm{tr}\,\Phi^2 \rangle}}$
0.4
0.2
0
−0.2
−0.4
0
200
400
600
800
1000
Φ
$\frac{S}{g_2/(4g_4)}$
−0.400
−0.405
−0.410
−0.415
−0.420
0
200
400
600
800
1000
Φ
C
0.65
0.60
0.55
0.50
0.45
0
200
400
600
800
1000
Φ
SDI/N^2
1.02
1.01
1.00
0.99
0.98
0
200
400
600
800
1000
Φ



$\frac{\mathrm{tr}\,\Phi / N}{\sqrt{g_2/(2g_4)}}$
0.10
0.05
0.00
−0.05
−0.10
200
400
600
800
1000
MC steps
$\frac{|S_N|}{g_2/(4g_4)}$
0.40
0.38
0.36
0.34
0.32
200
400
600
800
1000
MC steps


$C_{PP}^{(th)}$
0.5
0.4
0.3
0.2
0.1
−4
−2
0
2
4
g_2
S
1
0
−1
−2
−3
−4
−5
0.0
0.5
1.0
1.5
2.0
2.5
g_2
C
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.0
0.5
1.0
1.5
2.0
2.5
g_2
E_k
0.5
0.4
0.3
0.2
0.1
0.0
0.5
1.0
1.5
2.0
2.5
g_2


χ
0.8
0.6
0.4
0.2
0.0
0.5
1.0
1.5
2.0
2.5
g2
M
1.0
0.8
0.6
0.4
0.2
0.0
0.5
1.0
1.5
2.0
2.5
g2
χ±
0.8
0.6
0.4
0.2
0.0
0.5
1.0
1.5
2.0
2.5
g2
M±
0.5
0.4
0.3
0.2
0.1
0.0
0.5
1.0
1.5
2.0
2.5
g2



$$S_N \ni \mathrm{tr}\left(g_r|R|\Phi^2\right) \Rightarrow \delta g_2^{tp} \leq 16 g_r \qquad \delta g_2^{tp} \geq 16 g_r \Leftarrow \Phi_R^2 = \frac{g_2\mathbb{1} - g_r|R|}{2g_4}$$

$$\searrow \qquad \swarrow$$

$$\boxed{\delta g_2^{tp} = 16 g_r}$$

- triple-point shift:

$$\delta g_2^{\mathrm{tp}} \propto g_r \quad \Longrightarrow \quad \delta G_2^{\mathrm{tp}} \propto N g_r \quad \Longrightarrow \quad \boxed{G_2^{\mathrm{tp}} \sim N \to \infty}$$

- mass shift ($\delta m^2_{\mathrm{ren}}$: e.g. Franchino-Viñas and Pisani '14):

$$m^2_{\mathrm{bare}} + \delta m^2_{\mathrm{ren}} = m^2_{\mathrm{ren}} \leftrightarrow m^2_{\mathrm{ph}}$$

$$\delta m^2_{\mathrm{ren}} = \frac{\lambda}{12\pi(1+\Omega^2)} \log \frac{\Lambda^2_{\mathrm{max}}\theta}{\Omega} \quad \Longrightarrow \quad \boxed{\left|\delta G_2^{\mathrm{ren}}\right| \sim \log N < G_2^{\mathrm{tp}}}$$

- $\lambda\phi^4_\star$-renormalization (Grosse and Wulkenhaar '03):

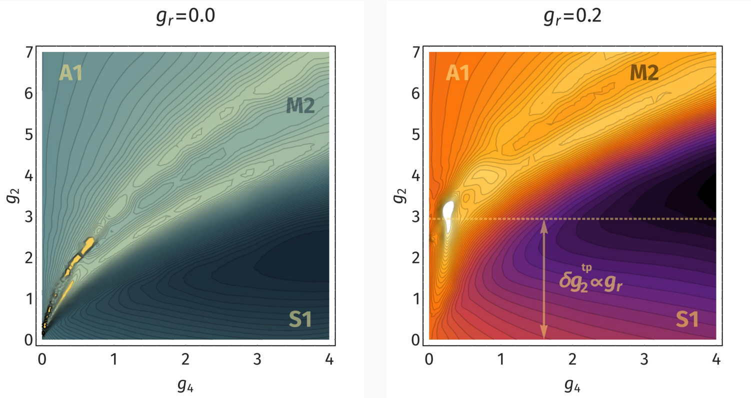
$$\Lambda^2_{\max} \sim N, \quad g_r \sim \frac{1}{\log^2 N} \to 0 \quad \Rightarrow \quad \mathrm{GW} \to \lambda\phi^4_\star$$

- triple-point shift:

$$\delta g_2^{\mathrm{tp}} \propto g_r \quad \Rightarrow \quad \delta G_2^{\mathrm{tp}} \propto N g_r \quad \Rightarrow \quad \boxed{G_2^{\mathrm{tp}} \sim \frac{N}{\log^2 N} \to \infty}$$

- mass shift:

$$\delta m^2_{\mathrm{ren}} = \frac{\lambda}{12\pi(1+\Omega^2)} \log \frac{\Lambda^2_{\max}\theta}{\Omega} \quad \Rightarrow \quad \boxed{\left|\delta G_2^{\mathrm{ren}}\right| \sim \log N < G_2^{\mathrm{tp}}}$$



$$\left|\delta G_2^{\mathrm{ren}}\right| < G_2^{\mathrm{tp}} \quad \Longrightarrow$$ bare model stays in S1-phase!

Distribution hunt

$$S_N(\Phi) = N\ \mathrm{tr}\Big(\Phi \widehat{K} \Phi - g_r R \Phi^2 - g_2 \Phi^2 + g_4 \Phi^4\Big)$$

$$\Phi = U \Lambda U^\dagger, \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$$

$$S_N(\Lambda, U) = N\ \mathrm{tr}\Big((U\Lambda U^\dagger)\widehat{K}(U\Lambda U^\dagger) - g_r R U \Lambda^2 U^\dagger - g_2 \Lambda^2 + g_4 \Lambda^4\Big)$$

$$Z = \int [d\Phi]\, e^{-S_N}, \qquad d\Phi = dU \prod_{i=1}^{N} d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2$$

$$\Delta^2(\Lambda) = \prod_{i<j} (\lambda_i - \lambda_j)^2$$

$$S_{\text{eff}}(\Lambda) = N\ \text{tr}\Big(\textbf{?} - g_2\Lambda^2 + g_4\Lambda^4\Big) - \sum_{m\neq n} \log|\lambda_m - \lambda_n|$$

$$\rho(\lambda) = \frac{1}{N}\sum_{i=1}^{N}\delta(\lambda-\lambda_i), \qquad \int d\lambda\, \rho(\lambda) = 1$$

$$\delta S_{\text{eff}} = 0 \quad \Longrightarrow \quad \textbf{?} - g_2\lambda + 2g_4\lambda^3 = \int_{\text{support}} d\lambda'\, \frac{\rho(\lambda')}{\lambda-\lambda'}$$

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example: $\rho(\lambda) = \dfrac{g_4(r^2+2\lambda^2) - g_2}{\pi}\sqrt{r^2-\lambda^2}$

- From here: neglecting the kinetic term to isolate the R-effects

$$Z = \int [d\Phi]\, e^{-S_N} = \int [d\Lambda]\, \Delta^2(\Lambda)\, e^{-N\,\mathrm{tr}\left(-g_2\Lambda^2 + g_4\Lambda^4\right)} \int [dU]\, e^{g_r N\,\mathrm{tr}\left(URU^\dagger\Lambda^2\right)}$$

$$I = \int_{U(N)} [dU]\; e^{t\,\mathrm{tr}\left(AUBU^\dagger\right)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} I_n \qquad\qquad I_n = \int [dU]\; \mathrm{tr}^n\left(AUBU^\dagger\right)$$

$$I = \exp\left(-\sum_{n=1}^{\infty} \frac{t^n}{n!} S_n\right)$$

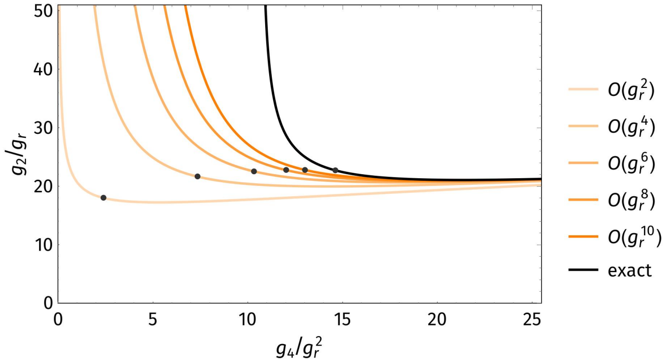
$$S_1 = -I_1 \quad\Rightarrow\quad S_2 = S_1^2 - I_2 \quad\Rightarrow\quad S_3 = -S_1^3 + 3S_1S_2 - I_3 \quad\Rightarrow\quad \ldots$$

	I_1	I_2	I_3	I_4	I_5	I_6
terms	1	4	36	576	14,400	518,400
time*	0.2 s	0.3 s	1 s	19 s	10 min	21 h

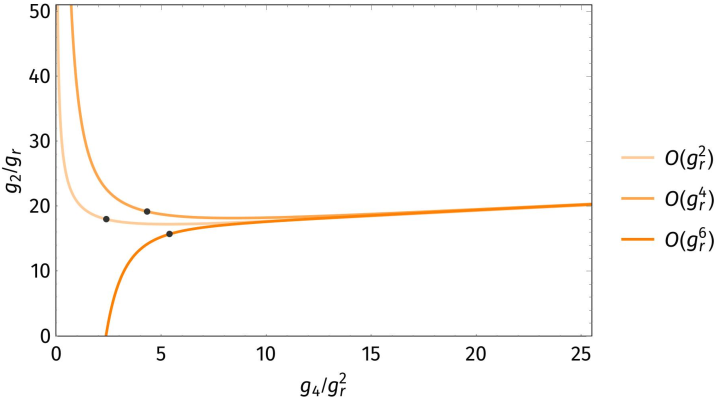
*Ryzen 7, 16 GB RAM + 20 GB swap, Ubuntu 20, Python

- Fukuda M, König R, Nechita I. RTNI—A symbolic integrator for Haar-random tensor networks. J. Phys. A: Math. Theor. 52 (2019)
- *Mathematica* or *Python*, up to 20 U-matrices
- additional Python code for collecting the repeating terms
- in the end, I_n has p_n^2 terms: 1, 4, 9, 25, 49, 121, ...

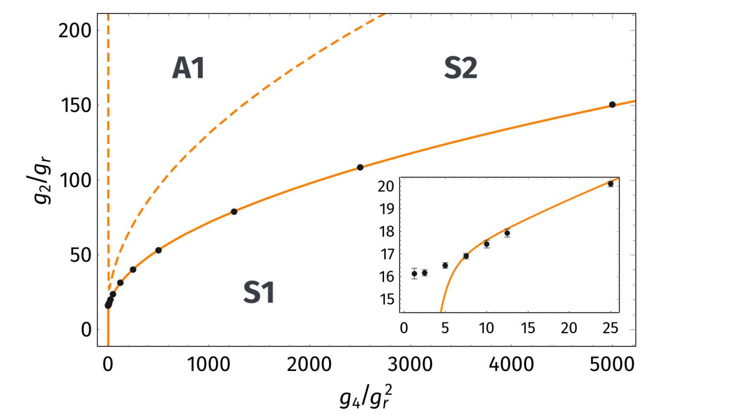
$$
\begin{aligned}
S_{\text{eff}}(\Lambda) = &-(g_2 - 8g_r) N \operatorname{tr}\Lambda^2 + \left(g_4 - \frac{(8g_r)^2}{6}\right) N \operatorname{tr}\Lambda^4 + \frac{(8g_r)^2}{6} \operatorname{tr}^2\Lambda^2 \\
&+ \frac{(8g_r)^4}{180} N \operatorname{tr}\Lambda^8 + \frac{(8g_r)^4}{60} \operatorname{tr}^2\Lambda^4 - \frac{(8g_r)^4}{45} \operatorname{tr}\Lambda^6 \operatorname{tr}\Lambda^2 \\
&- \frac{(8g_r)^6}{2835} N \operatorname{tr}\Lambda^{12} + \frac{2(8g_r)^6}{945} \operatorname{tr}\Lambda^2 \operatorname{tr}\Lambda^{10} - \frac{(8g_r)^6}{189} \operatorname{tr}\Lambda^4 \operatorname{tr}\Lambda^8 + \frac{2(8g_r)^6}{567} \operatorname{tr}^2\Lambda^6 \\
&- \log \Delta^2(\Lambda)
\end{aligned}
$$



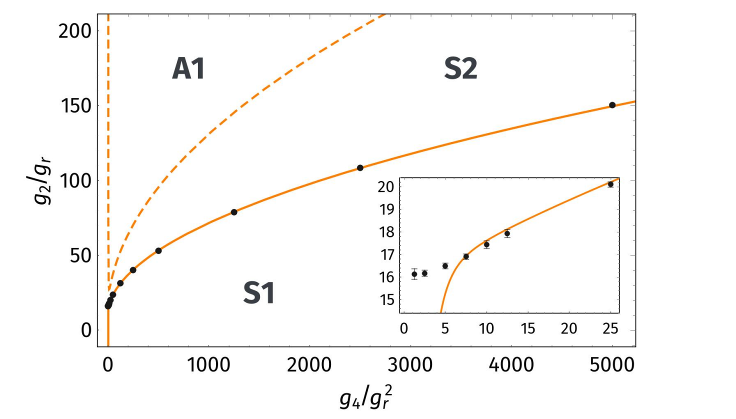
$$g_2^{\text{exact}} = 2\sqrt{g_4 - \frac{1}{6}(8g_r)^2} + 8g_r + \frac{\frac{1}{3}(8g_r)^2}{\sqrt{g_4 - \frac{1}{6}(8g_r)^2}}$$



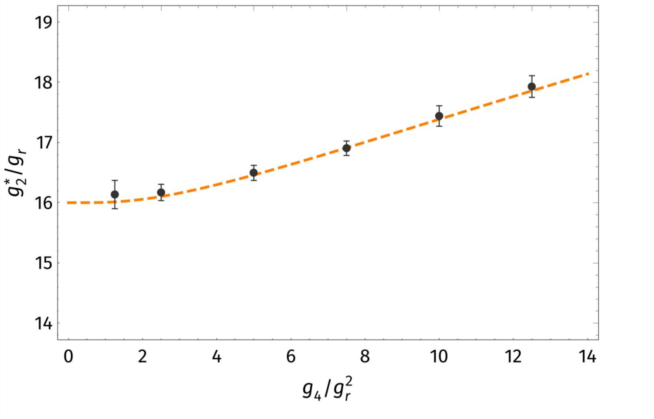
g_2/g_r
50
40
30
20
10
0
0
5
10
15
20
25
g_4/g_r^2
$O(g_r^2)$
$O(g_r^4)$
$O(g_r^6)$



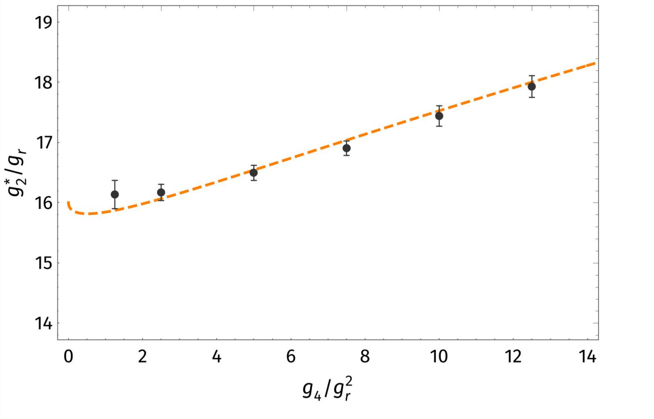
- solid orange line — 6th order analytical expression
- black dots — HMC simulation data extrapolated to $N \to \infty$



$$g_2 = 2\sqrt{g_4} + 8g_r + \frac{1}{6}\frac{(8g_r)^2}{\sqrt{g_4}} + \frac{1}{240}\frac{(8g_r)^4}{g_4\sqrt{g_4}} - \frac{1}{1344}\frac{(8g_r)^6}{g_4^2\sqrt{g_4}}$$



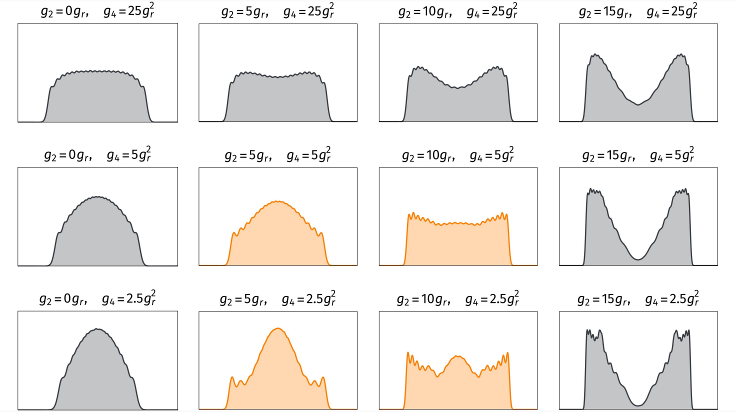
guess 1: $$g_2 = \frac{16g_r}{1 - \exp\left(-\frac{8g_r}{\sqrt{g_4}}\right)} = 8g_r\left(1 + \coth\frac{8g_r}{2\sqrt{g_4}}\right)$$



guess 2: $$g_2 = 4\sqrt{g_4} + 12 g_r + \frac{16 g_r^2}{\sqrt{g_4}}\left(1 - \exp\left(-\frac{3\sqrt{g_4}}{4 g_r}\right)\right) - \frac{g_4}{2 g_r}\,\psi_1\!\left(\frac{\sqrt{g_4}}{4 g_r}\right)$$

100 EUR prize to whoever first finds a simple nonperturbative expression for the transition line that has the correct expansion and starting point, and agrees with the numerical results.

New behavior near $g_4 = 0$?



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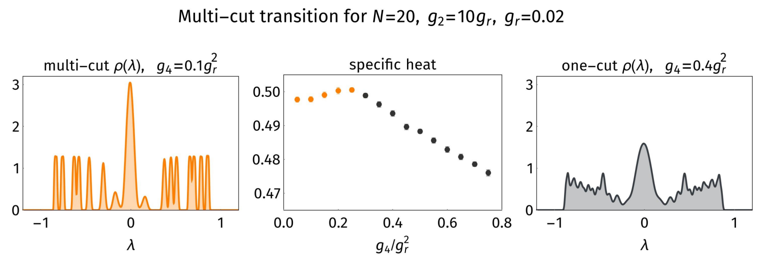


Beyond GW: A gauge model

- start with the 3-dim action on ϵ-tHA (Burić, Grosse, Madore '10):

$$S_{\mathrm{YM}} = \frac{1}{16g^2}\,\mathrm{tr}\big(F(*F) + (*F)F\big)$$

$$\downarrow$$

"compactification" to $z = 0$

$$\downarrow$$

$$S_{\mathrm{YM}} = \frac{1}{2}\mathrm{tr}\Big(\frac{1-\epsilon^2}{g^2}F_{12}^2 + (D\phi)^2 + (5-\epsilon^2)\mu^2\phi^2 - \\ - 2(1-\epsilon^2)\frac{\mu}{g}F_{12}\phi - 4\epsilon F_{12}\phi^2 + \epsilon^2\{P+gA,\phi\}^2\Big)$$

where

$$D_\alpha\phi = i[P_\alpha + gA_\alpha, \phi] \qquad F_{12} = ig[P_1, A_2] - ig[P_2, A_1] + ig^2[A_1, A_2]$$

- EOM:

$$\frac{1-\epsilon^2}{g}\mu F_{12} + 2\epsilon\{F_{12},\phi\} - (5-\epsilon^2)\mu^2\phi + D^\alpha D_\alpha\phi - \epsilon^2\{X^\alpha,\{X_\alpha,\phi\}\} = 0$$

$$\frac{(1-\epsilon^2)\epsilon^{\alpha\beta}}{g^2}D_\beta(F_{12}-\mu g\phi) - 2\epsilon\epsilon^{\alpha\beta}\{D_\beta\phi,\phi\} - i[D^\alpha\phi,\phi] + \epsilon^2\{\{X^\alpha,\phi\},\phi\} = 0$$

$$\text{where}\quad X_\alpha = P_\alpha + gA_\alpha$$

- two classical vacua, both with $S = 0$:

1.	$A_1 = 0$	$A_2 = 0$	$\phi = 0$
2.	$A_1 = -\frac{\mu^2 y}{\epsilon g}$	$A_2 = +\frac{\mu^2 x}{\epsilon g}$	$\phi = \frac{\mu}{\epsilon g}$

- $\mathrm{tr}\, X = \mathrm{tr}\, Y = 0$, eigenvalues of both signs therefore stripe phase?

- $\phi\phi$-sector:

$$\epsilon^2 \pi^3 \mu^2 g^2 \Lambda_{\max}^2 \int \phi \, \frac{1}{\square} \, \phi$$

$$(8/\epsilon^2 - 14 + \epsilon^2)\pi^3 \mu^4 g^2 \log \Lambda_{\max} \int \phi \, \frac{1}{\square^2} \, \phi$$

- AA-sector:

$$-\frac{(1-\epsilon^2)\pi^3 \mu^2 g^2}{\epsilon} \Lambda_{\max} \log \Lambda_{\max} \int A^\alpha \frac{1}{\square} A_\alpha$$

- nonrenormalizable (Burić, Nenadović, Prekrat '16)

- Chern-Simons term (Burić, Grosse, Madore '10):

$$S_{\mathrm{CS}} = -\frac{\alpha\mu}{3}\,\mathrm{tr}\left(-(3-\epsilon^2)\left(F_{12}-\frac{\mu^2}{\epsilon}\right)\phi + 2\epsilon X^\alpha X_\alpha\left(\phi - \frac{\mu}{2\epsilon}\right)\right)$$

- no trivial vacuum
- striped vacuum:

$$S_{\mathrm{CS}} = 0 \qquad \Big| \qquad S_{\mathrm{YM}} + S_{\mathrm{CS}} = 0$$

- translationally invariant nontrivial vacuum @ $\epsilon = 1$, $\alpha = 6$:

$$A_1 = A_2 = 0 \qquad \phi = \mu \qquad \Big| \qquad S_{\mathrm{YM}} + S_{\mathrm{CS}} < 0$$

- X, Y implementation for $\epsilon < 1$?

$$[X, Y] = i\epsilon\left(\mathbb{1} - Z\right) \qquad [X, Z] = i\epsilon\left\{Y, Z\right\} \qquad [Y, Z] = -i\epsilon\left\{X, Z\right\}$$

$$\Downarrow$$

$$\left[X/\sqrt{\epsilon}, Y/\sqrt{\epsilon}\right] = i\left(\mathbb{1} - Z\right) \qquad \cancelto{0}{[X, Z] = i\epsilon\left\{Y, Z\right\}} \qquad \cancelto{0}{[Y, Z] = -i\epsilon\left\{X, Z\right\}}$$

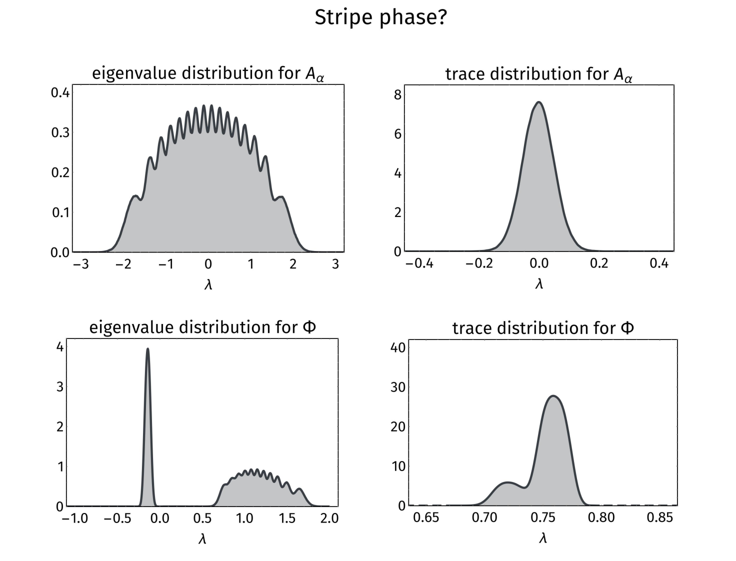
- $Z \to 0$ implementation problem

$$F_{12} = i[P_1, A_2] - i[P_2, A_1] + ig[A_1, A_2] \propto [X, Y]$$

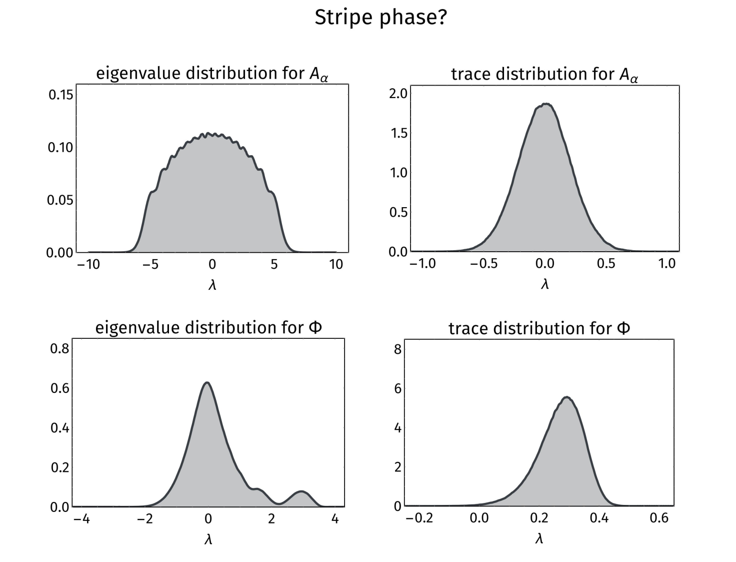
$$\Downarrow$$

$$F_{12}^2 \propto \mathbb{1} - 2Z + Z^2, \quad O(N) \text{ vs. } O(N^2)$$

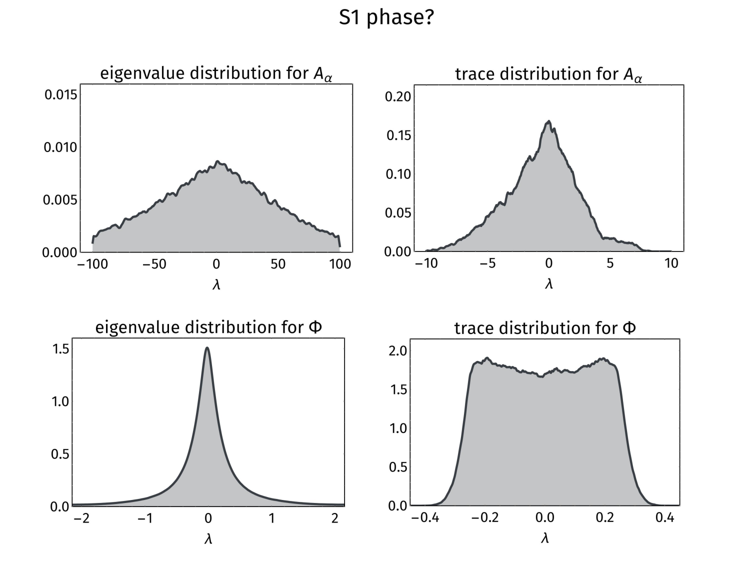
First numerical results — $\epsilon = 0.1$, $N = 16$



First numerical results — $\epsilon = 0.7$, $N = 16$



First numerical results — $\epsilon = 1.0$, $N = 16$



Phase/renormalizability connection

space	model	mass shift	triple point	start. phase	renorm.
$\mathbb{R}^2_\theta$, tHA	$\lambda\phi^4_\star$	UV/IR	0	M2	NO
$\mathbb{R}^2_\theta$, tHA	GW	$\log N$	N	S1	YES
$\mathbb{R}^2_\theta$, tHA	$\lambda\phi^4_{\mathrm{GW}}$	$\log N$	$N \big/ \log^2 N$	S1	YES
ϵ-tHA	$U(1)$		two phases?	M2?	NO

- hypothesis: no stripe-phase ↔ renormalizability
- connection is demonstrated between the GW model renormalizability and its phase structure

- analytical front:
    - kinetic-term effects
- numerical front:
    - gauge model, Chern-Simons
    - 4-dim GW
- methodology front:
    - method extendable to other existing models
    - GPU utilization (CUDA)
    - ML utilization
    - bootstrap
- speculations: astrophysics, superfluid dark matter

- numerical simulation of phase transitions could tell us in advance about the renormalization properties of new models

We can support participants form CEEPUS affiliated countries and institutions.

The deadline for registrations for financial support is 30. 6. 2026 at 12:00.

The support will be 73 EUR per day of attended activities.

Deadline for other registrations is 31. July 2026 at 12:00.

The End

Thank you for your attention.